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Optimization for Buckling of Composite Sandwich Plates

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Consider a composite sandwich plate with balanced or unbalanced anisotropic composite laminated faces and an ideally orthotropic core. A closed-form solution of the buckling load is derived for the composite sandwich plates consisting of cross-ply symmetric laminate faces with all edges simply supported. In addition, a general scheme is developed for the composite sandwich plates with arbitrary sandwich layups, boundary, and loading conditions. By combining the present buckling analysis with Powell's conjugate direction method, an optimal algorithm is established to find the optimal sandwich layup in resisting biaxial compression, as well as in-plane shear loads.

I. Introduction

SANDWICH plate is one of the commonly used structural constructions. The reason for its use is largely the great bending stiffness resulting from the load carrying faces being separated by the core. The low specific weight is another attractive feature. These two major advantages can be further enhanced by the introduction of fiber-reinforced composite laminates for the faces. When such structural construction is subjected to biaxial compression, as well as in-plane shear loads, there are several ways it may fail. One obvious way is by overstressing. Before overstressing, the structure can buckle in various ways. Thus, it is desirable to develop a general procedure for the buckling analysis of the composite sandwich plates.

From the viewpoint of material anisotropy, early researchers on the sandwich plates consisting of laminated faces for the most part assumed that the principal material axes are parallel to the geometric axes, i.e., are orthotropic.[1,2] Later, some researchers considered the analysis of anisotropic sandwich plates by the Rayleigh–Ritz method.[3,4] We consider the most general case of anisotropic sandwich plates by applying the mathematical model proposed by Hwu and Hu.[5] Using that model, a more general numerical scheme is developed, which considers arbitrary sandwich layups (symmetric or unsymmetric, balanced or unbalanced, othotropic or anisotropic), boundary conditions (simply supported, clamped, or others), and loading conditions (uniaxial, or biaxial, or shear loading). Moreover, an analytical solution is derived for the special case of sandwich plates consisting of cross-ply symmetric laminate faces with all edges simply supported.

As to the optimization for the buckling, much work has been done for the laminated composite plates.[6,7] However, very few studies have considered the composite sandwich plates where the materials of the core and faces are usually very different and the effect of transverse shear deformation is important. To provide an optimum composite sandwich plate in resisting the biaxial compression or shear load, an optimum algorithm is developed by combining the present buckling analysis with Powell's[8] zero-order conjugate direction method. In our study, the fiber orientations of the face laminas are chosen to be the design variables, and the buckling load is the objective function. The results show that the optimal arrangement is always in the form of $[(\theta/-\theta)_n/\text{core}]_s$ regardless of its boundary and loading conditions; i.e., maximum buckling load occurs when the faces of sandwich plates are angle-ply laminates.

II. Buckling Analysis of Composite Sandwich Plates

A mathematical model for the buckling analysis of composite sandwich plates was proposed by Hwu and Hu.[5] According to their model, the stress resultant–strain relations, the finite deformation kinematic relations, and the equilibrium equations for the buckled plates can be expressed as follows.

A. Stress Resultant–Strain Relations

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{1a}$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = c \begin{Bmatrix} G_{xz}\gamma_{xz} \\ G_{yz}\gamma_{yz} \end{Bmatrix} \tag{1b}$$

where (N_x, N_y, N_{xy}) and (M_x, M_y, M_{xy}) represent the in-plane loadings and bending moments, respectively, which are almost contributed by the faces; (Q_x, Q_y) are the transverse shear forces, which are undertaken by the core; $(\epsilon_x, \epsilon_y, \gamma_{xy})$ and $(\kappa_x, \kappa_y, \kappa_{xy})$ are the midplane strain and curvature of the sandwich plates; and γ_{xz} and γ_{yz} are the transverse shear strain of the x–z and y–z planes. Here c is the thickness of the core, and G_{xz} and G_{yz} are transverse shear moduli in x–z and y–z planes. A_{ij}, B_{ij}, and D_{ij} are the extensional, coupling, and bending stiffnesses, respectively, which are related to the location z_k and the transformed reduced stiffnesses $\bar{Q}_{ij}$ of each lamina (see Fig. 1) as

$$A_{ij} = \sum_{k=1}^{n} (\bar{Q}_{ij})_k (z_k - z_{k-1}) \qquad B_{ij} = \frac{1}{2}\sum_{k=1}^{n} (\bar{Q}_{ij})_k \left(z_k^2 - z_{k-1}^2\right)$$
$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} (\bar{Q}_{ij})_k \left(z_k^3 - z_{k-1}^3\right) \tag{1c}$$

Unlike the classical lamination theory in which A_{ij}, B_{ij}, and D_{ij} are calculated based on the coordinate where $z = 0$ is the middle surface of the laminate, here the plane $z = 0$ is located on the midsurface of the core.

B. Finite Deformation Kinematic Relations

The relations are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{2a}$$

$$\kappa_x = \frac{\partial}{\partial x}\left(\gamma_{xz} - \frac{\partial w}{\partial x}\right) \qquad \kappa_y = \frac{\partial}{\partial y}\left(\gamma_{yz} - \frac{\partial w}{\partial y}\right)$$
$$\kappa_{xy} = \frac{\partial}{\partial y}\left(\gamma_{xz} - \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(\gamma_{yz} - \frac{\partial w}{\partial y}\right) \tag{2b}$$

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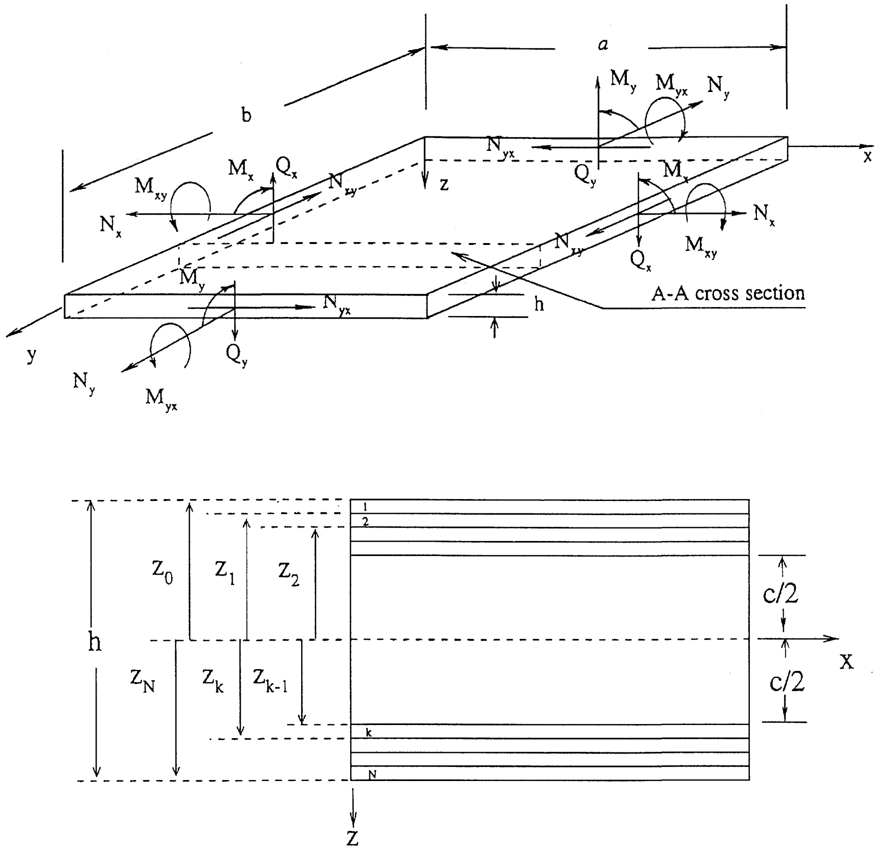

A–A cross section

Fig. 1 Loading and geometry of a composite sandwich plate.

where u, v, and w are the midplane displacements of the x, y, and z directions, respectively.

C. Equilibrium Equations for the Buckled Plates

The equations are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \tag{3}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x \qquad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y$$

By using Eqs. (1) and (2), the five equilibrium equations for the buckled composite sandwich plates given in Eq. (3) can be written in terms of five unknowns u, v, w, γ_{xz}, and γ_{yz}. Theoretically, the solutions can be found by solving these five coupled partial differential equations with the proper boundary conditions. However, due to the mathematical infeasibility, only very few special cases can be solved analytically. Most of the problems should be solved numerically. In the following sections, we present an analytical solution for the composite sandwich plates with cross-ply symmetric laminate faces. After that, a general numerical procedure is established to solve the general buckling problems of composite sandwich plates. The accuracy of the numerical procedure is then verified by the analytical solution.

III. Analytical Solutions for Special Problems

A special composite sandwich plate with cross-ply symmetric laminate faces is treated. For this kind of sandwich plate, $A_{16} = A_{26} = D_{16} = D_{26} = 0$, $B_{ij} = 0$, $i, j = 1, 2, 6$, by Eq. (1c). If one considers a small deflection, the nonlinear terms in Eq. (2a) can be neglected. Under these special considerations, the in-plane and plate bending problems are uncoupled and can be treated individually. For plate bending problems, only the last three equations of Eq. (3) should be considered. By substituting Eqs. (1) and (2) into Eq. (3) with the preceding considerations, we obtain the governing equations as follows:

$$\left(N_x \frac{\partial^2}{\partial x^2} + N_y \frac{\partial^2}{\partial y^2} + 2N_{xy}\frac{\partial^2}{\partial x \partial y}\right) w + cG_{xz}\frac{\partial \gamma_{xz}}{\partial x} + cG_{yz}\frac{\partial \gamma_{yz}}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left[D_{11}\frac{\partial^2}{\partial x^2} + (D_{12} + 2D_{66})\frac{\partial^2}{\partial y^2}\right] w - \left[D_{11}\frac{\partial^2}{\partial x^2} + D_{66}\frac{\partial^2}{\partial y^2} - cG_{xz}\right]\gamma_{xz} - (D_{12} + D_{66})\frac{\partial^2 \gamma_{yz}}{\partial x \partial y} = 0 \tag{4}$$

$$\frac{\partial}{\partial y}\left[D_{22}\frac{\partial^2}{\partial y^2} + (D_{12} + 2D_{66})\frac{\partial^2}{\partial x^2}\right] w - \left[D_{22}\frac{\partial^2}{\partial y^2} + D_{66}\frac{\partial^2}{\partial x^2} - cG_{yz}\right]\gamma_{yz} - (D_{12} + D_{66})\frac{\partial^2 \gamma_{xz}}{\partial x \partial y} = 0$$

in which the plane loadings N_x, N_y, and N_{xy} are treated as known quantities to the bending problem and are assumed to be unchanged during bending.

Consider a rectangular sandwich plate of sides a and b, with cross-ply symmetric laminate faces, simply supported on all edges, and subjected to biaxial compression $N_x = -P$ and $N_y = -kP$. The boundary conditions for simply supported edges can be described as

$$\begin{aligned} &w = 0, \quad M_y = 0, \quad \gamma_{xz} = 0, \quad y = 0, b \\ &w = 0, \quad M_x = 0, \quad \gamma_{yz} = 0, \quad x = 0, a \end{aligned} \tag{5}$$

Here an edge stiffener is considered to prevent shear strain. If no forces parallel to the edges of the plate are applied to prevent shear strain, $\gamma_{xz} = 0$ and $\gamma_{yz} = 0$ should be replaced by $M_{xy} = 0$. By the fourth and fifth equations in Eqs. (1a) and the first two of Eqs. (2b) and $D_{16} = D_{26} = 0$, $B_{ij} = 0$, $i, j = 1, 2, 6$, the boundary conditions (5) can now be rewritten as

$$w = 0, \qquad \gamma_{xz} = 0, \qquad \frac{\partial}{\partial y}\left(\gamma_{yz} - \frac{\partial w}{\partial y}\right) = 0, \qquad y = 0, b$$
$$w = 0, \qquad \gamma_{yz} = 0, \qquad \frac{\partial}{\partial x}\left(\gamma_{xz} - \frac{\partial w}{\partial x}\right) = 0, \qquad x = 0, a \tag{6}$$

which will be satisfied if one assumes

$$w = W \sin(m\pi x/a) \sin(n\pi y/b)$$
$$\gamma_{xz} = \mathbf{\Gamma}_x \cos(m\pi x/a) \sin(n\pi y/b) \tag{7}$$
$$\gamma_{yz} = \mathbf{\Gamma}_y \sin(m\pi x/a) \cos(n\pi y/b)$$

Substituting Eq. (7) into Eq. (4) with $N_x = -P$, $N_y = -kP$, and $N_{xy} = 0$, we obtain a system of three simultaneous linear equations in three unknowns W, $\mathbf{\Gamma}_x$, and $\mathbf{\Gamma}_y$. Nontrival solutions exist only when the determinant of the coefficients vanishes, which leads to a solution for the buckling load of P. The final simplified result is

$$P_{cr} = \frac{1 + \eta}{[1 + k(an/bm)^2](1 + \eta_x + \eta_y + \eta_o)} P_o \tag{8a}$$

where

$$\eta = \frac{\pi^2 D^*}{c}\left[\frac{1}{G_{yz}}\left(\frac{m}{a}\right)^2 + \frac{1}{G_{xz}}\left(\frac{n}{b}\right)^2\right]$$
$$\eta_x = \frac{\pi^2}{cG_{xz}}\left[D_{11}\left(\frac{m}{a}\right)^2 + D_{66}\left(\frac{n}{b}\right)^2\right]$$
$$\eta_y = \frac{\pi^2}{cG_{yz}}\left[D_{66}\left(\frac{m}{a}\right)^2 + D_{22}\left(\frac{n}{b}\right)^2\right] \qquad \eta_o = \frac{D^* P_o m^2 n^2}{a^2 c^2 G_{xz} G_{yz}}$$
$$P_o = \frac{\pi^2 a^2}{m^2}\left[D_{11}\left(\frac{m}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\frac{mn}{ab}\right)^2 + D_{22}\left(\frac{n}{b}\right)^4\right]$$
$$D^* P_o = \frac{\pi^2 a^2}{m^2}\left[D_{11} D_{66}\left(\frac{m}{a}\right)^4 + [D_{11} D_{22} - D_{12}(D_{12} + 2D_{66})]\left(\frac{mn}{ab}\right)^2 + D_{22} D_{66}\left(\frac{n}{b}\right)^4\right] \tag{8b}$$

in which only the lowest value of P_{cr} is usually of any importance. However, it is not clear which values of m and n result in the lowest critical buckling load. Values of m and n should be determined computationally. A detail discussion about the m and n values will be given in example 1 of Sec. VI.

Note that because the formulation listed in Sec. II is for composite sandwich plates of which the core thickness is usually greater than the face thickness, the result obtained in Eq. (8) is not suitable for the case when c is relatively smaller than the face thickness. However, by letting the terms concerning the core properties vanish (i.e., $\eta = \eta_x = \eta_y = \eta_o = 0$), not by letting c or G_{xz} or G_{yz} be zero, Eq. (8) will be reduced to $P_{cr} = [1 + k(an/bm)^2]^{-1} P_o$. Under uniaxial loading $k = 0$; the result may be further reduced to $P_{cr} = P_o$, which is identical to that presented in Ref. 9 for the cases of composite laminates (without core).

IV. Numerical Techniques for General Problems

As already mentioned, the equilibrium equation (3) together with the constitutive relations (1) and the kinematic relations (2) provide five equations in terms of five unknowns u, v, w, γ_{xz}, and γ_{yz}. Therefore, if one can find a set of functions u, v, w, γ_{xz}, and γ_{yz} satisfying these five equations, as well as the boundary and loading conditions, the solution to the postbuckling stage may be found. To determine the buckling load, the stage immediately after the buckling is usually considered. At that stage, the deflection is small and the nonlinear terms in Eq. (2a) are negligible in comparison with the remaining terms. Although the solution procedure just stated for the buckling load is straightforward, it is usually mathematically infeasible because they are nonlinear coupled partial differential equations. Only the special cases, like that shown in Sec. III, can be solved analytically. In engineering practice, any kind of sandwich layup and boundary and loading conditions may occur. Especially for optimum structural design, we need a general scheme to deal with all of the possible combinations. Thus, it is necessary for us to find an alternative approach. We choose the Rayleigh–Ritz method to find the buckling load of composite sandwich plate with arbitrary boundary conditions, which is a convenient procedure for determining solutions by the principle of minimum potential energy.

Consider a rectangular composite sandwich plate of sides a and b. The total potential energy Π of this sandwich plate consists of the strain energy U due to bending, stretching, and shearing and the potential energy $-W$ of the external forces, i.e.,

$$\Pi = U - W \tag{9a}$$

where

$$U = \frac{1}{2}\int_0^b\int_0^a (M_x\kappa_x + M_y\kappa_y + M_{xy}\kappa_{xy} + N_x\epsilon_x + N_y\epsilon_y + N_{xy}\gamma_{xy} + Q_x\gamma_{xz} + Q_y\gamma_{yz})\,\mathrm{d}x\,\mathrm{d}y$$
$$W = \frac{1}{2}\int_0^b\int_0^a \left[\bar{N}_x\left(\frac{\partial w}{\partial x}\right)^2 + \bar{N}_y\left(\frac{\partial w}{\partial y}\right)^2 + 2\bar{N}_{xy}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right]\mathrm{d}x\,\mathrm{d}y \tag{9b}$$

$\bar{N}_x$, $\bar{N}_y$, and $\bar{N}_{xy}$ denote external in-plane forces, which should be distinguished from the internal in-plane forces N_x, N_y, and N_{xy}. Unlike the special cases discussed in Sec. III, where N_x, N_y, and N_{xy} are treated as known quantities and are assumed to be unchanged during bending, in general, N_x, N_y, and N_{xy}, like all of the other force components, cannot be solved separately.

Substituting Eqs. (1a), (1b), (2a), and (2b) with the nonlinear terms neglected (inasmuch as only the bifurcation buckling is concerned in our analysis, which may be studied by considering small deflection) into Eq. (9), the total potential energy may be expressed as follows:

$$\Pi = \frac{1}{2}\int_0^b\int_0^a [f(x, y) - \lambda g(x, y)]\,\mathrm{d}x\,\mathrm{d}y \tag{10}$$

where $f(x, y)$ is the strain energy density, the explicit expression of which expressed by u, v, w, γ_{xz}, and γ_{yz} is provided in the Appendix, and $g(x, y)$ is a function related to the work done by the external force $\bar{N}_x = \lambda$, $\bar{N}_y = \lambda k_1$, and $\bar{N}_{xy} = \lambda k_2$. That is,

$$g(x, y) = \left(\frac{\partial w}{\partial x}\right)^2 + k_1\left(\frac{\partial w}{\partial y}\right)^2 + 2k_2\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \tag{11}$$

We now choose a solution for the deformation u, v, w, γ_{xz}, and γ_{yz} in the form of series containing undetermined parameters U_{ij}, V_{ij}, W_{ij}, Φ_{ij}, and Ψ_{ij} $(i, j = 1, 2, \ldots)$, i.e.,

$$u = \sum_{i=1}^{m}\sum_{j=1}^{n} U_{ij} u_{xi}(x) u_{yj}(y) \qquad v = \sum_{i=1}^{m}\sum_{j=1}^{n} V_{ij} v_{xi}(x) v_{yj}(y)$$
$$w = \sum_{i=1}^{m}\sum_{j=1}^{n} W_{ij} w_{xi}(x) w_{yj}(y) \qquad \gamma_{xz} = \sum_{i=1}^{m}\sum_{j=1}^{n} \Phi_{ij}\phi_{xi}(x)\phi_{yi}(y)$$
$$\gamma_{yz} = \sum_{i=1}^{m}\sum_{j=1}^{n} \Psi_{ij}\psi_{xi}(x)\psi_{yi}(y) \tag{12}$$

The deformation so selected must satisfy the geometric boundary conditions. The static boundary conditions need not be fulfilled.

After employing the selected solution, the potential energy Π may now be expressed in terms of U_{ij}, V_{ij}, W_{ij}, Φ_{ij}, and Ψ_{ij}. This demonstrates that U_{ij}, V_{ij}, W_{ij}, Φ_{ij}, and Ψ_{ij} govern the variation of the potential energy. For the potential energy to be a minimum at equilibrium,

$$\frac{\partial \Pi}{\partial U_{ij}} = 0, \quad \frac{\partial \Pi}{\partial V_{ij}} = 0, \quad \frac{\partial \Pi}{\partial \Phi_{ij}} = 0, \quad \frac{\partial \Pi}{\partial \Psi_{ij}} = 0$$
$$\frac{\partial \Pi}{\partial W_{ij}} = 0, \qquad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \tag{13}$$

which yield a system of $5mn$ algebraic equations. Symbolically, the $4mn$ algebraic equations obtained from the first four of Eqs. (13) may be written as

$$A\alpha = B\beta \tag{14a}$$

where

$$\alpha = \{U_{11}, \ldots, U_{mn}, V_{11}, \ldots, V_{mn}, \Phi_{11}, \ldots, \Phi_{mn}, \Psi_{11}, \ldots, \Psi_{mn}\}^T$$
$$\beta = \{W_{11}, \ldots, W_{mn}\}^T \tag{14b}$$

A and B are $4mn \times 4mn$ and $4mn \times mn$ coefficient matrices, respectively. Solving Eq. (14) by Gauss–Jordan elimination technique, U_{ij}, V_{ij}, Φ_{ij}, and Ψ_{ij} may then be expressed in terms of W_{ij}, i.e.,

$$\alpha = A^{-1}B\beta \tag{15}$$

Similarly, the last equation of Eqs. (13) leads to

$$\frac{\partial \Pi}{\partial W_{ij}} = \frac{1}{2}\int_0^b\int_0^a \left(\frac{\partial f}{\partial W_{ij}} - \lambda\frac{\partial g}{\partial W_{ij}}\right) dx\, dy = 0$$
$$i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \tag{16a}$$

which may also be written as

$$C\alpha + D\beta - \lambda G\beta = 0 \tag{16b}$$

where $C\alpha + D\beta$ and $G\beta$ are, respectively, the symbolic expressions for

$$\frac{1}{2}\iint \frac{\partial f}{\partial W_{ij}} dx\, dy \quad \text{and} \quad \frac{1}{2}\iint \frac{\partial g}{\partial W_{ij}} dx\, dy$$

Substituting Eq. (15) into Eq. (16b) and rearranging, we obtain the following eigenvalue problem:

$$F\beta = \lambda G\beta \tag{17a}$$

where

$$F = CA^{-1}B + D \tag{17b}$$

F and G are both $mn \times mn$ real symmetric matrices. From Eq. (17a), numerous eigenvalues λ_{mn} may be found; the lowest value of λ_{mn} will be the buckling load for the considered composite sandwich plates.

Note that in the preceding derivation, A, B, C, D, and G are shown only symbolically without providing their explicit expressions because they are lengthy and tedious. A relative simple example is given for the purpose of illustration:

$$\frac{\partial g}{\partial W_{ij}} = 2\sum_{p=1}^{m}\sum_{q=1}^{n}(w_{xi,x}w_{yj}w_{xp,x}w_{yq} + k_1 w_{xi}w_{yj,y}w_{xp}w_{yq,y}$$
$$+ k_2 w_{xi,x}w_{yj}w_{xp}w_{yq,y} + k_2 w_{xi}w_{yj,y}w_{xp,x}w_{yq}) \tag{18}$$

A computer program has been developed by following the procedures just stated. At the input status, we provide some special functions u_{xi}, v_{xi}, w_{xi}, ϕ_{xi}, and ψ_{xi} and u_{yj}, v_{yj}, w_{yj}, ϕ_{yj}, and ψ_{yj} for some special boundary conditions, such as

$$\begin{aligned} u_{xi} &= \cos(i\pi x/a), & u_{yj} &= \sin(j\pi y/b) \\ v_{xi} &= \sin(i\pi x/a), & v_{yj} &= \cos(j\pi y/b) \\ w_{xi} &= \sin(i\pi x/a), & w_{yj} &= \sin(j\pi y/b) \\ \phi_{xi} &= \cos(i\pi x/a), & \phi_{yj} &= \sin(j\pi y/b) \\ \psi_{xi} &= \sin(i\pi x/a), & \psi_{yj} &= \cos(j\pi y/b) \end{aligned} \tag{19}$$

for all edges simply supported. In the case of all edges clamped, the functions $u_{xi}, \ldots, \psi_{yj}$ are chosen to be the same as Eq. (19) except for w_{xi} and w_{yj}, which are chosen to be

$$w_{xi} = 1 - \cos(2i\pi x/a), \qquad w_{yj} = 1 - \cos(2j\pi y/b) \tag{20}$$

For all of the other boundary conditions where $u_{xi}, \ldots, w_{yj}$ are not provided, the users should input these functions by considering the satisfaction of the geometric boundary condition. The remaining input information necessary for the implementation of the computer program comprise the material properties of faces and core, fiber orientation of each lamina, size of the plate, and loading ratios (k_1 and k_2).

V. Buckling Strength Optimization

In an optimum structural design, weight is the most commonly used objective function. Limitations on the maximum stresses, displacements, or buckling strength are usually treated as constraints. Plate size and material properties are then chosen to be the design variables. In this paper, if the material properties, number of face lamina, plate size, and so forth are all predetermined, the weight of the composite sandwich plate will not be changed during the design process. This means that the weight is not appropriate to be the objective function. Moreover, because our main concern is the influence of the fiber orientation on the buckling strength, the best choice for our objective function should be the buckling strength and that for the design variables should be the fiber orientation of each lamina and no constraint equations are required. Therefore, the mathematical formulation can now be written as follows:

$$\text{Maximize} \quad P_{cr}(\theta_1, \theta_2, \ldots, \theta_n) \tag{21}$$

where P_{cr} is the buckling strength and θ_i ($i = 1, 2, \ldots, n$) is the fiber orientation of the ith lamina of the faces.

From the numerical analysis given in the last section, we see that the objective function is a complicated function of design variables. Under this condition, the numerical calculation of the gradient and

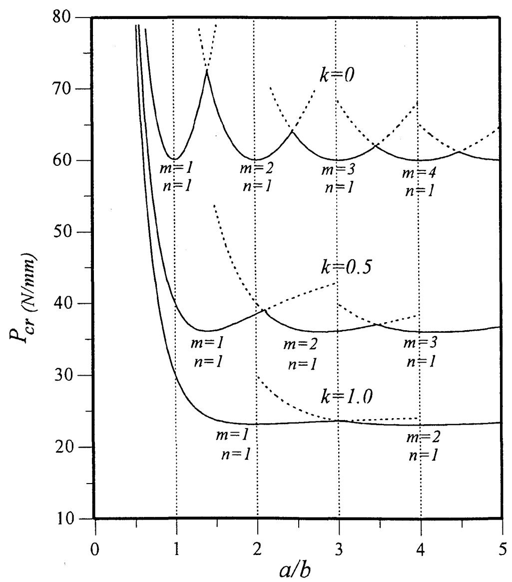

Fig. 2 Variation of the lowest critical buckling load P_{cr} with respect to the aspect ratio a/b and the biaxial compression ratio k; b = 1000 mm.

Hessian matrix will be costly. Therefore, the zero-order methods for the unconstrained optimization problem are selected. Among various zero-order methods, Powell's[8] conjugate direction method is one of the most efficient and reliable methods and provides the search directions for our optimization process. Along each search direction, the successive quadratic method[10] is applied to determine the distance in the search direction that will maximize P_{cr} as much as possible.

Note that in the actual implementation of the optimum techniques just described it is seldom possible to ensure that the absolute optimum design is found in the first trial. Hence, it is usual to start the program from several different initial designs.

VI. Results and Discussion

In the following, the correctness of the buckling analysis is checked in examples 1 and 2, and the results of buckling optimization are presented and discussed in example 3.

A. Example 1: Buckling of a Composite Sandwich Plate with Cross-Ply Symmetric Laminate Faces

The purpose of this example is to compare the numerical results obtained by using the Rayleigh–Ritz method presented in Sec. IV with the analytical solutions found in Sec. III. The data used for comparison are as follows: 1) graphite/epoxy layered faces with $E_{11} = 181$ GPa, $E_{22} = 10.3$ GPa, $G_{12} = 7.17$ GPa, $\nu_{12} = 0.28$, and $t_{ply} = 0.125$ mm and 2) aluminum honeycomb core with $G_{xz} = 0.146$ GPa, $G_{yz} = 0.0904$ GPa, and $c = 10$ mm. Here, E, G, and ν are Young's modulus, shear modulus, and Poisson's ratio, respectively. The subscript 1 denotes the fiber direction, and the subscript 2 denotes the transverse direction. The thickness of lamina ply and core are t_{ply} and c, respectively. The results are shown in Table 1, in which only four terms ($i = 1, 2$ and $j = 1, 2$) are used to get a convergent value for the numerical approximation (except for the last one, which uses six terms), and the values of m and n corresponding to the lowest critical buckling load of the analytical solution are shown in Fig. 2 (except for the last two, which are shown in Table 1). The numerical values in Table 1 show that the analytical and numerical solutions are exactly the same, which is expected because the forms of the trial functions chosen in Eq. (19) happen to be the exact solutions given in Eq. (7).

Table 1 Buckling of a composite sandwich plate ([(0 deg/90 deg)$_2$/honeycomb]$_s$) with all edges simply supported

a, mm	b, mm	k $(= N_y/N_x)$	P_{cr}, N/mm, numerical[a]	P_{cr}, N/mm, analytical[b]
500	1000	0	114.194	114.194
1000	1000	0	60.0532	60.0534
2000	1000	0	60.0519	60.0534
500	1000	0.5	101.505	101.506
1000	1000	0.5	40.0355	40.0356
2000	1000	0.5	38.4320	38.4322
500	1000	1	91.3549	91.3552
1000	1000	1	30.0266	30.0267
2000	1000	1	23.0592	23.0593
100	1000	0.5	940.653	940.627[c]
200	1000	1	417.688	417.680[d]

[a]Numerical results obtained by using Rayleigh–Ritz method in Sec. IV.
[b]Analytical solution obtained by using Eq. (8) of which the m and n values corresponding to the lowest critical buckling load are shown in Fig. 2, except the last two rows, which are noted by superscripts c and d.
[c]Lowest critical buckling load is obtained when $m = 1$ and $n = 2$.
[d]Lowest critical buckling load is obtained when $m = 1$ and $n = 3$.

From Fig. 2, we also observe that no matter how P_{cr} varies with respect to a/b and k, in most cases $n = 1$. If $k = 0$, then $m \approx \sqrt[4]{(D_{22}/D_{11})}(a/b)$, which is consistent with the m value corresponding to the lowest P_o. Because all values of n appear in the numerator of P_o, the necessary value to get the lowest P_o is $n = 1$. By setting $n = 1$ and $\partial P_o/\partial m = 0$ (treating m as a real number), we get $m = \sqrt[4]{(D_{22}/D_{11})}(a/b)$. Thus, $m = \sqrt[4]{(D_{22}/D_{11})}(a/b)$ if $\sqrt[4]{(D_{22}/D_{11})}(a/b)$ is an integer. Otherwise, m will be an integer near $\sqrt[4]{(D_{22}/D_{11})}(a/b)$. To know when n will not be unity, we present two cases in the last two rows of Table 1 that occur when $k \neq 0$ and a/b is small.

B. Example 2: Buckling of a Composite Sandwich Plate with Single-Layered Faces

In the case that the sandwich faces are not cross-ply laminates, the closed-form solution [Eq. (8)] cannot be applied, and a numerical technique such as the one proposed in Sec. IV becomes necessary. To check the correctness of the present numerical scheme, comparison is made with results presented by Rao[3] and by Kim and Hong[4] for a sandwich plate with single-layered faces where the fiber orientation θ is varied from 0 to 90 deg. The data used for comparison are taken from these references and are as follows: 1) carbon/epoxy layered faces with $E_{11} = 229$ GPa, $E_{22} = 13.35$ GPa, $G_{12} = 5.25$ GPa, $\nu_{12} = 0.315$, and $t_{ply} = 0.2$ mm and 2) aluminum honeycomb core with $G_{xz} = 0.146$ GPa, $G_{yz} = 0.0904$ GPa, and $c = 10$ mm. The results presented in Table 2 show that the present solutions are slightly different from the others, within 0.79%, which is acceptable. For this special case, which is calculated for the purpose of comparison, it seems there is no difference between our proposed method and the others. In the next example, several different conditions are considered for the optimization of the sandwich buckling, which will show that our formulation is more general than the others.

Table 2 Buckling of a composite sandwich plate ([θ/honeycomb]$_s$) all edges simply supported under uniaxial compressive load N_x, with $a = b = 225$ mm

θ, deg \ P_{cr}, N/mm	Rao[3]	Kim and Hong[4]	Present
0	423.8	427.6	424.2
10	428.7	432.4	429.3
20	443.5	447.0	444.3
30	459.5	463.4	460.7
40	468.7	472.4	470.1
45	——	467.0	464.9
50	——	437.9	435.2
60	356.6	360.1	357.9
70	285.6	288.4	286.9
80	233.6	235.6	234.8
90	213.8	215.6	214.9

C. Example 3: Buckling Optimization of a Composite Sandwich Plate

The material properties and the core thickness used in this example are the same as those of example 1. Each face is composed of four plies whose thickness is $t_{ply} = 0.125$ mm. As shown in Table 3, several different conditions are considered in this example to show the generality of the present numerical scheme. They are different plate sizes (varying a and b), different loading conditions (varying k), and different boundary conditions: all edges simply supported (ss); clamped along edges $x = 0, a$, simply supported along edges

Table 3 Optimal fiber orientations for the buckling of a composite sandwich plate[a]

			θ/P_{cr}, deg N/mm			
a, mm	b, mm	$k(= N_y/N_x)$	ss	cs	sc	cc
1000	500	0	44/352.66	39/404.28	36/444.73	31/494.21
1000	1000	0	45/99.932	18/157.01	42/152.85	36/179.75
1000	2000	0	0/50.310	0/146.69	0/51.488	0/147.65
1000	1000	0.5	45/66.617	22/112.26	54/107.42	38/123.60
1000	1000	1.0	45/49.963	29/85.774	61/79.453	40/92.933
1000	1000	2.0	45/33.308	37/56.240	63/51.207	43/61.504

[a]All of the optimal arrangements are $[(\theta/-\theta)_2/\text{honeycomb}]_s$.

$y = 0, b$ (cs); simply supported along edges $x = 0, a$, clamped along edges $y = 0, b$ (sc); and all edges clamped (cc). At the beginning of the search for the optimal fiber orientation for the buckling load, each fiber orientation of the face laminas is assumed to be an independent design variable. After actual implementing the zero-order Powell's conjugate direction method for multiple-variables unconstrained nonlinear optimization, we find that the optimal arrangement is always in the form of $[(\theta/-\theta)_n/\text{core}]_s$. That is, maximum buckling load occurs when the faces of the sandwich plates are angle-ply laminates. Table 3 also shows that 1) P_{cr} decreases when b/a or $k(= N_y/N_x)$ increases and 2) $(P_{cr})_{cc} \geq (P_{cr})_{sc\text{ or }cs} \geq (P_{cr})_{ss}$. These behaviors are expected and are the same as the isotropic plates or composite laminates.

VII. Concluding Remarks

By applying the Rayleigh–Ritz method, a general procedure is established for the determination of the buckling load of composite sandwich plates with arbitrary sandwich layups and boundary and loading conditions. In our own design computer program, the users can choose their own trial functions $u_{xi}, \ldots, w_{yj}$ [Eq. (12)] to suit their boundary conditions. After they input these functions, all of the mathematical operations, such as differentiation, will be done automatically by computers. Moreover, for users' convenience, we input some special functions for some special boundary conditions such as the one given in Eqs. (19) and (20).

In the buckling optimization part, the most curious thing for us is how to prove theoretically that the optimal arrangement of the composite sandwich plates resisting biaxial compressive loads is always in the form of $[(\theta/-\theta)_n/\text{core}]_s$, regardless of the boundary and loading conditions.

Appendix: Explicit Expression for Strain Energy Density $f(x,y)$

$$
\begin{aligned}
f(x,y) = {} & A_{11}\left(\frac{\partial u}{\partial x}\right)^2 + 2A_{16}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + A_{66}\left(\frac{\partial u}{\partial y}\right)^2 \\
& + A_{22}\left(\frac{\partial v}{\partial y}\right)^2 + 2A_{26}\left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) + A_{66}\left(\frac{\partial v}{\partial x}\right)^2 \\
& + 2A_{12}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right) + 2A_{16}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right) + 2A_{26}\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) \\
& + 2A_{66}\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + 2B_{11}\left(\frac{\partial u}{\partial x}\frac{\partial \gamma_{xz}}{\partial x}\right) + 2B_{16}\left(\frac{\partial u}{\partial x}\frac{\partial \gamma_{xz}}{\partial y}\right) \\
& + 2B_{16}\left(\frac{\partial u}{\partial y}\frac{\partial \gamma_{xz}}{\partial x}\right) + 2B_{66}\left(\frac{\partial u}{\partial y}\frac{\partial \gamma_{xz}}{\partial y}\right) + 2B_{12}\left(\frac{\partial u}{\partial x}\frac{\partial \gamma_{yz}}{\partial y}\right) \\
& + 2B_{16}\left(\frac{\partial u}{\partial x}\frac{\partial \gamma_{yz}}{\partial x}\right) + 2B_{26}\left(\frac{\partial u}{\partial y}\frac{\partial \gamma_{yz}}{\partial y}\right) + 2B_{66}\left(\frac{\partial u}{\partial y}\frac{\partial \gamma_{yz}}{\partial x}\right) \\
& + 2B_{26}\left(\frac{\partial v}{\partial y}\frac{\partial \gamma_{xz}}{\partial y}\right) + 2B_{16}\left(\frac{\partial v}{\partial x}\frac{\partial \gamma_{xz}}{\partial x}\right) + 2B_{12}\left(\frac{\partial v}{\partial y}\frac{\partial \gamma_{xz}}{\partial x}\right) \\
& + 2B_{66}\left(\frac{\partial v}{\partial x}\frac{\partial \gamma_{xz}}{\partial y}\right) + 2B_{22}\left(\frac{\partial v}{\partial y}\frac{\partial \gamma_{yz}}{\partial y}\right) + 2B_{26}\left(\frac{\partial v}{\partial y}\frac{\partial \gamma_{yz}}{\partial x}\right) \\
& + 2B_{26}\left(\frac{\partial v}{\partial x}\frac{\partial \gamma_{yz}}{\partial y}\right) + 2B_{66}\left(\frac{\partial v}{\partial x}\frac{\partial \gamma_{yz}}{\partial x}\right) + D_{11}\left(\frac{\partial \gamma_{xz}}{\partial x}\right)^2 \\
& + 2D_{16}\left(\frac{\partial \gamma_{xz}}{\partial x}\frac{\partial \gamma_{xz}}{\partial y}\right) + D_{66}\left(\frac{\partial \gamma_{xz}}{\partial y}\right)^2 + 4D_{26}\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x \partial y}\right) \\
& + 2D_{12}\left(\frac{\partial \gamma_{xz}}{\partial x}\frac{\partial \gamma_{yz}}{\partial y}\right) + 2D_{16}\left(\frac{\partial \gamma_{xz}}{\partial x}\frac{\partial \gamma_{yz}}{\partial x}\right) \\
& + 2D_{26}\left(\frac{\partial \gamma_{xz}}{\partial y}\frac{\partial \gamma_{yz}}{\partial y}\right) + 2D_{66}\left(\frac{\partial \gamma_{xz}}{\partial y}\frac{\partial \gamma_{yz}}{\partial x}\right) + D_{22}\left(\frac{\partial \gamma_{yz}}{\partial y}\right)^2 \\
& + 2D_{26}\left(\frac{\partial \gamma_{yz}}{\partial y}\frac{\partial \gamma_{yz}}{\partial x}\right) + D_{66}\left(\frac{\partial \gamma_{yz}}{\partial x}\right)^2 + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \\
& + D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) + 4D_{16}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial y}\right) \\
& + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 - 2B_{11}\left(\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) - 2B_{12}\left(\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) \\
& - 4B_{16}\left(\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right) - 2B_{16}\left(\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) - 2B_{26}\left(\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial y^2}\right) \\
& - 4B_{66}\left(\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right) - 2B_{26}\left(\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) - 4B_{66}\left(\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right) \\
& - 2B_{12}\left(\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial x^2}\right) - 2B_{22}\left(\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y^2}\right) - 4B_{26}\left(\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right) \\
& - 2B_{16}\left(\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) - 2D_{11}\left(\frac{\partial \gamma_{xz}}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) - 2D_{12}\left(\frac{\partial \gamma_{xz}}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) \\
& - 4D_{16}\left(\frac{\partial \gamma_{xz}}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right) - 2D_{16}\left(\frac{\partial \gamma_{xz}}{\partial y}\frac{\partial^2 w}{\partial x^2}\right) \\
& - 2D_{26}\left(\frac{\partial \gamma_{xz}}{\partial y}\frac{\partial^2 w}{\partial y^2}\right) - 4D_{66}\left(\frac{\partial \gamma_{xz}}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right) \\
& - 2D_{12}\left(\frac{\partial \gamma_{yz}}{\partial y}\frac{\partial^2 w}{\partial x^2}\right) - 2D_{16}\left(\frac{\partial \gamma_{yz}}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) \\
& - 4D_{26}\left(\frac{\partial \gamma_{yz}}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right) - 2D_{22}\left(\frac{\partial \gamma_{yz}}{\partial y}\frac{\partial^2 w}{\partial y^2}\right) \\
& - 2D_{26}\left(\frac{\partial \gamma_{yz}}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) - 4D_{66}\left(\frac{\partial \gamma_{yz}}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right) \\
& + c\,G_{xz}\gamma_{xz}^2 + c\,G_{yz}\gamma_{yz}^2
\end{aligned}
$$

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